Noise Induced Oscillation in Solutions of Stochastic Delay Differential Equations

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ABSTRACT: This paper studies the oscillatory properties of solutions of linear scalar stochastic delay differential equations with multiplicative noise. It is shown that such noise will induce an oscillation in the solution whenever there is negative feedback from the delay term. The zeros of the process are a countable set; the solution is differentiable at each zero, and the zeros are simple. The addition of such noise does not alter the positivity of solutions when there is positive feedback.
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1. INTRODUCTION

Delay differential equations are widely used to model systems in ecology, physics, and economics. Very often, interest focusses on solutions of such equations which are oscillatory, as these could plausibly reflect cyclic motion of a system around an equilibrium. Over the last thirty years, an extensive theory of oscillatory solutions of deterministic equations has developed. However, the effect that random perturbations of Itô type might have on the existence— creation or destruction— of oscillatory solutions of delay differential equations seems, at present, to be absent from the literature. In this paper, we study the oscillation of solutions about the equilibrium position of (autonomous) linear scalar stochastic delay differential equations.

In the deterministic (and stochastic) case, the oscillations in the solutions of first order delay differential equations are generated by the delayed argument, as first order *ordinary* differential equations (and scalar stochastic differential equations) do not possess oscillatory solutions about their equilibrium. In particular, the presence of noise does not induce an oscillation about the equilibrium, if the equilibrium is a strong solution of the stochastic differential equation. In order to observe an oscillation in a scalar stochastic delay differential equation, therefore, we must consider the joint effects of the delayed argument and the stochastic perturbation.

The main result of the paper is that while nonoscillatory solutions can exist in the deterministic case when there is a small negative feedback from the delay term, this can never happen under the presence of a multiplicative noise. On the other hand, when there is positive feedback from the delay term, and the initial function always has the same sign, the solution retains its sign, even in the presence of multiplicative noise. These results motivate the title of the paper: oscillation is induced in a previously nonoscillatory system by the presence of noise.

We also show that the zero set of the solution is a countable set of points, and that the process, which is in general non-differentiable, is differentiable at its zeros; in fact each zero is simple.

2. MOTIVATION AND BACKGROUND MATERIAL

In this section, we introduce the notion of oscillation of a stochastic process (subsection 2.1) and indicate mechanisms which cause the oscillation of solutions of ordinary and stochastic differential and delay differential equations (subsection 2.2). We also refer to those results from the theory of oscillation of solutions of deterministic delay differential equations which we require for our analysis, and sketch how those results enable us to prove the oscillation of solutions of stochastic delay differential equations (in subsection 2.3).

2.1. Oscillation of stochastic processes

We say that a non-trivial (i.e. $y(t) = 0$ for all $t \geq t_1$ for some $t_1 \geq t_0$ is excluded) continuous function $y : [t_0, \infty) \to \mathbb{R}$ is *oscillatory* if the set

$$Z_y = \{t \geq t_0 \,:\, y(t) = 0\}$$

satisfies $\sup Z_y = \infty$. A function which is not oscillatory is called *nonoscillatory.* We extend these notions to stochastic processes in the following intuitive manner: a stochastic process $(X(t,\omega))_{t\geq t_0}$ defined on a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ with continuous sample paths is said to be *almost surely oscillatory* (a.s. oscillatory hereafter) if there exists $\Omega^* \subseteq \Omega$ with $\mathbb{P}[\Omega^*] = 1$ such that for all $\omega \in \Omega^*$, the path $X(\cdot, \omega)$ is oscillatory. A stochastic process is *a.s. nonoscillatory* if there exists $\Omega^* \subseteq \Omega$ with $\mathbb{P}[\Omega^*] = 1$ such that for all $\omega \in \Omega^*$, the path $X(\cdot, \omega)$ is nonoscillatory.

2.2. Oscillation of scalar linear evolutions

One of the important characteristics of the sample paths of solutions of (Itô) scalar stochastic differential equations is that there are, in general, no points at which the path is differentiable.

Consequently, it might be thought that the oscillation of the solutions of such equations about *equilibrium* is a generic phenomenon. However, as for scalar differential equations, one must distinguish between cases which are perturbed by terms which vanish at equilibrium, and those which do not.

To motivate this observation, let us first consider the ordinary scalar differential equation $x'(t) = ax(t)$, which admits the equilibrium solution $x(t) \equiv 0$. For $x(0) \neq 0$, solutions of this equation do not oscillate about the equilibrium solution. Consider now two perturbations of this equation, namely

$$x'(t) = ax(t) + bx(t), \quad t \geq 0, \tag{1}$$

for $b \neq 0$, and

$$x'(t) = ax(t) + p(t), \quad t \geq 0, \tag{2}$$

where p is a continuous non-trivial T-periodic function which satisfies $\int_0^T p(s)\,ds = 0$.

If $x(0) \neq 0$, solutions of (1) do not oscillate about the equilibrium solution zero. However, for equation (2), all solutions oscillate about zero, provided $a < 0$. The perturbation in (1) preserves $x(t) \equiv 0$ as equilibrium solution, while $x(t) \equiv 0$ is not a solution of (2). In this case therefore, the oscillation arises from a perturbation which does not vanish at equilibrium.

The same phenomenon can be seen for linear stochastic differential equations. We consider two (Itô) stochastic perturbations of $x'(t) = ax(t)$, namely

$$dX(t) = aX(t)\,dt + \sigma X(t)\,dB(t) \tag{3}$$

and

$$dX(t) = aX(t)\,dt + \sigma dB(t), \tag{4}$$

where σ is a positive constant, and $(B(t))_{t\geq 0}$ is standard one–dimensional Brownian motion.

Again, (3) has $X(t) \equiv 0$ as an equilibrium solution, and, for $X(0) \neq 0$, (3) has the solution

$$X(t) = X(0) \cdot \exp((a - \sigma^2/2)t + \sigma B(t)),$$

which does not oscillate around the equilibrium solution. The solution of (4) is given by

$$X(t) = e^{at}X(0) + \sigma e^{at} \int_0^t e^{-as} dB(s).$$

Whenever $a < 0$, one can use the martingale time change theorem and the law of the iterated logarithm to prove that

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} = \frac{\sigma}{\sqrt{2|a|}}, \quad \liminf_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} = -\frac{\sigma}{\sqrt{2|a|}}, \quad a.s.,$$

so the continuity of the sample paths of (4) ensures that every solution of (4) is a.s. oscillatory. In this case, $X(t) \equiv 0$ is not an equilibrium solution.

Therefore, we see that oscillation for scalar linear differential equations relies on non-equilibrium preserving perturbations, and that no oscillation can occur about equilibrium if the perturbation preserves the equilibrium of the unperturbed system. Indeed, the presence of "noise" alone is not sufficient to cause oscillation about an equilibrium for scalar stochastic differential equations.

Of course, it is possible for systems of linear differential equations to oscillate about an equilibrium solution, if the system (or equation) is of order two or greater. This is well-known for deterministic systems, but stochastic systems also exhibit this phenomenon. Stochastic oscillators are considered, e.g. in Mao, 1997. As an example, consider the system

$$dX_1(t) = \sigma X_2(t)\, dB(t), \quad dX_2(t) = -\sigma X_1(t)\, dB(t).$$

The solution of this equation is

$$\begin{aligned} X_1(t) &= e^{\frac{\sigma^2}{2}t} \left(\cos(\sigma B(t)) X_1(0) + \sin(\sigma B(t)) X_2(0) \right), \\ X_2(t) &= e^{\frac{\sigma^2}{2}t} \left(-\sin(\sigma B(t)) X_1(0) + \cos(\sigma B(t)) X_2(0) \right). \end{aligned}$$

Consequently, if the initial conditions are deterministic, we see that both X_1 and X_2 oscillate about zero, almost surely.

The presence of a delay term in a scalar linear delay differential equation is, however, sufficient to induce the oscillation about zero of its solutions under certain conditions. This holds even when zero is a solution of the problem. Indeed, although the equation $x'(t) = bx(t)$ does not have oscillatory solutions for $b < 0$, it transpires for $b < 0$ that all non-trivial solutions of

$$x'(t) = bx(t - \tau), \quad t \geq 0 \tag{5}$$

are oscillatory provided $-be\tau > 1$, while nonoscillatory solutions of (5) exist if $-be\tau \leq 1$ (this result can be found in Proposition 1.3.2 in Gopalsamy, 1992. Therefore, all solutions are oscillatory if the delay is sufficiently long, but nonoscillatory solutions can still exist for small delay (or small intensity b). In the case when $b > 0$, solutions are positive (and therefore nonoscillatory) if the initial function defined on $[-\tau, 0]$ is strictly positive. It is now natural to ask: what are the oscillatory properties of solutions of a stochastic (but equilibrium preserving) perturbation of (5), e.g.,

$$dX(t) = bX(t - \tau)\, dt + \sigma X(t)\, dB(t). \tag{6}$$

It is this problem, and closely related problems, which we seek to address in this paper. It transpires for $b < 0$ that all solutions of (6) are a.s. oscillatory, so it is no longer possible to have nonoscillatory solutions, even for small delay (or small feedback intensity b), and an oscillation is induced by the noise perturbation. When there is positive feedback from the delay term (for $b > 0$), solutions are a.s. positive, provided the initial function on $[-\tau, 0]$ is strictly positive, as for those of (5); therefore, noise does not appear to induce oscillation in solutions in the presence of positive feedback from the past.

2.3. Statement of the problem; background theory

This paper studies the a.s. oscillatory and nonoscillatory nature of solutions of the scalar stochastic delay differential equation

$$dX(t) = (aX(t) + bX(t-\tau(t)))\,dt + \sigma X(t)\,dB(t) \tag{7a}$$

$$X(t) = \psi(t), \quad -\overline{\tau} \le t \le 0, \tag{7b}$$

where $\tau(t) \le \overline{\tau}$ is a continuous function satisfying some additional conditions, and ψ is a continuous function in $C([-\overline{\tau}, 0])$. We will frequently remark on the case when $\tau(t) = \tau$, i.e. problem (7) has a constant delay.

The solution of (7) is a stochastic process $(X(t,\omega))_{t\ge t_0}$ defined on a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. We will denote deterministic functions by small letters, stochastic processes by capital letters, for ease of notation we will sometimes suppress the dependence on ω. Often we are interested in comparing the oscillatory properties of the solutions of this problem to those of the corresponding deterministic one. For (7a) we compare with

$$x'(t) = ax(t) + bx(t-\tau(t)), \quad t \ge 0. \tag{8}$$

with the same initial function ψ.

Our strategy for proving the existence of a.s. oscillatory solutions involves writing the solution of equation (7a) in terms of the (continuously differentiable) solution of a scalar random delay differential equation of the form

$$Y'(t) = -P(t)Y(t-\tau(t)) \tag{9}$$

where P is a random, non-negative function. (In the deterministic case a reduction of (8) to a pure delay equation can be easily obtained by setting $x(t) = y(t)e^{at}$ to find $y(t) = be^{-a\tau(t)}y(t-\tau(t))$.) Then one can invoke (on a path-wise basis) some of the extensive existing deterministic theory of oscillatory solutions of delay differential equations (we refer the reader to the monographs of Gopalsamy, 1992 or Ladde et al., 1987. A short summary of the results we use follows.

The following result on oscillatory solutions can be found in Theorem 2.1.3] in Ladde et al., 1987, it is a special case of Theorem 2 in Staikos and Stavroulakis, 1977.

Proposition 1. *Suppose that $p(\cdot)$ is a continuous, nonnegative function defined on $[t_0, \infty)$ which satisfies*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} p(s)\,ds > 1$$

where $g : [t_0, \infty) \to \mathbb{R}^+$ is a non-decreasing continuous function satisfying $g(t) < t$, $g(t) \to \infty$ as $t \to \infty$. Then all solutions of

$$y'(t) = -p(t)y(g(t)) \tag{10}$$

are oscillatory.

It is also possible to obtain nonoscillatory solutions of (10). The following result can be found in Gopalsamy, 1992 as Theorem 1.3.5.

Proposition 2. *Suppose that $p(\cdot)$ is a continuous, nonnegative function defined on $[t_0, \infty)$ which satisfies*

$$\int_{g(t)}^{t} p(s)\,ds \le \frac{1}{e}$$

for all $t > T$, where $g : [t_0, \infty) \to \mathbb{R}^+$ is a non-decreasing continuous function satisfying $g(t) < t$, $g(t) \to \infty$ as $t \to \infty$. Then (10) has a nonoscillatory solution.

We show in Section 3 that whenever g (with $g(t) = t - \tau(t)$) satisfies the hypotheses of Proposition 1, and $b < 0$, the solution of (7) is a.s. oscillatory for any choice of (deterministic) initial function ψ. Using Proposition 2, however, we remark that in the deterministic case equation (8) can have nonoscillatory solutions for sufficiently small negative b.

In the case of constant delay, the behaviour of the solutions of (8) can be characterized, using Proposition 1.2.10 in Gopalsamy, 1992, which goes back to a result of Driver et al., 1973.

Proposition 3. *Let $\tau(t) = \tau$, τ constant, $b < 0$, $-be^{-a\tau}e\tau < 1$. Then the solution of (8) satisfies*

$$\lim_{t\to\infty} x(t)e^{-\lambda_0 t} = \frac{1}{1+\lambda_0\tau}\left(\psi(0) + \lambda_0\tau \int_{-\tau}^{0} e^{-\lambda_0 s}\psi(s)\,ds\right),$$

where λ_0 is a real negative root of

$$\lambda - be^{-a\tau}e^{-\lambda\tau} = 0.$$

3. OSCILLATION OF SOLUTIONS

In this section, we establish the oscillatory and nonoscillatory properties of solutions of (7). We assume that $\tau(\cdot)$ is a bounded continuous function which satisfies

$$0 < \underline{\tau} < \tau(t) \leq \overline{\tau} < \infty, \tag{11}$$

together with

$$t \mapsto t - \tau(t) \text{ is non-decreasing.} \tag{12}$$

Introduce the process $(\Phi(t))_{t \geq -\overline{\tau}}$ which satisfies $\Phi(t) = 1$ for $t \in [-\overline{\tau}, 0]$ and $\Phi(t) = \exp((a - \sigma^2/2)t + \sigma B(t))$ for $t \geq 0$, i.e., it is the solution of the stochastic differential equation

$$d\Phi(t) = a\Phi(t)\, dt + \sigma\Phi(t)\, dB(t). \tag{13}$$

Also define for $t \geq -\overline{\tau}$ the process $Y(t) = X(t)/\Phi(t)$, where $X(t)$ is the solution of (7) (Y is well-defined, as Φ is a strictly positive process). Using (stochastic) integration by parts, we see that Y satisfies

$$Y(t) = Y(0) + \int_0^t b\, Y(s - \tau(s))\ \Phi(s - \tau(s))\ \Phi(s)^{-1}\, ds, \quad t \geq 0. \tag{14}$$

Consequently, $(X(t))_{t \geq 0}$ satisfies

$$X(t) = \Phi(t)\ \left(\psi(0) + \int_0^t b\ X(s - \tau(s))\ \Phi(s)^{-1}\, ds\right). \tag{15}$$

Returning to (14), we see that the continuity of the integrand on the right–hand side implies that $Y \in C^1((0, \infty); \mathbb{R})$. Differentiating (14) yields

$$Y'(t) = b\ \Phi(t - \tau(t))\ \Phi(t)^{-1} Y(t - \tau(t)). \tag{16}$$

Another way of obtaining a random differential equation to represent the solution of a stochastic differential equation is given in Lisei, 2001.

For $b > 0$ it is not difficult to show that all solutions of (7) are a.s. positive (and hence a.s. nonoscillatory) if the initial function $\psi \in C([-\overline{\tau}, 0], \mathbb{R}^+)$.

Proposition 4. *Let $b > 0$ and $\psi(t) > 0$ for all $-\overline{\tau} \leq t \leq 0$. Then (7) has an a.s. positive solution on $[0, \infty)$, where τ satisfies (11).*

Proof. Let $\Omega^* \subset \Omega$ be the almost sure event on which Y obeys (16). Let $\omega \in \Omega^*$. Since $Y(t, \omega) = \psi(t) > 0$ for all $t \in [-\overline{\tau}, 0]$, we may define $t^*(\omega) = \inf\{t \geq 0 : Y(t, \omega) = 0\}$. Indeed, as ψ is positive, $t^*(\omega) > 0$. By definition, we must have $Y'(t^*(\omega), \omega) \leq 0$. Since τ is a positive function $Y(t^*(\omega) - \tau(t^*(\omega)), \omega) > 0$, as $Y(t, \omega) > 0$ for $t < t^*(\omega)$. Therefore, as Φ is a positive process, and $b > 0$, we see from (16) that $Y'(t^*(\omega), \omega) > 0$, a contradiction. Therefore $Y(t, \omega) > 0$ for all $t \geq 0$. By construction, therefore, $X(t, \omega) > 0$ for all $t \geq 0$, and so X is almost surely positive. □

Figure 1: $dX(t) = \{0.35X(t) + 0.1X(t-1)\}dt + \sigma X(t)dW(t),\ \psi \equiv 1,\ \sigma = 0$ (left) and $\sigma = 0.8$ (right).

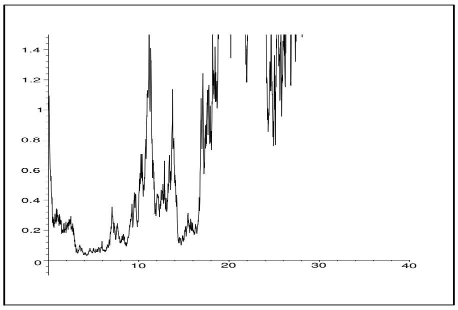

Figure 2: $dX(t) = \{0.35X(t) + 0.1X(t-1)\}dt + \sigma X(t)dW(t),\ \psi \equiv 1,\ \sigma = 0.8$, a zoom of the trajectory above.

Figures 1 and 2 illustrate the behaviour of solution trajectories under the conditions of Proposition 4.

We now show that solutions of (7) are a.s. oscillatory, no matter what the choice of ψ, whenever $b < 0$ and $\sigma \neq 0$.

Proposition 5. *Let $b < 0$, and τ satisfy (11) and (12). Then for any continuous function ψ, Equation (7) has an a.s. oscillatory solution on $[0, \infty)$.*

Proof. Since $Y(t) = X(t)/\Phi(t)$, the set

$$Z = \{t \geq 0 \ : \ X(t) = 0\}$$

satisfies $\sup Z = \infty$ a.s. if and only if the set

$$\tilde{Z} = \{t \geq 0 \ : \ Y(t) = 0\}$$

satisfies $\sup \tilde{Z} = 0$ a.s.. Define for $t \geq 0$, and $\omega \in \Omega$,

$$P(t, \omega) = -b\ \Phi(t - \tau(t), \omega)\ \Phi^{-1}(t, \omega),$$

Then, on $[0, \infty)$, $P(\cdot)$ is an a.s. positive, continuous function. Note now that Y satisfies

$$Y'(t, \omega) = -P(t, \omega)Y(t - \tau(t), \omega) \quad \text{for } t > 0. \tag{17}$$

Suppose there exists $\Omega^* \subset \Omega$ such that

$$\Omega^* = \{\omega \in \Omega \,:\, \limsup_{t\to\infty} \int_{t-\tau(t)}^{t} P(s, \omega)\, ds > 1\}, \quad \text{and} \quad \mathbb{P}[\Omega^*] = 1,$$

then as P and $g(t) = t - \tau(t)$ satisfy the conditions of Proposition 1, for each $\omega \in \Omega^*$, it follows that the path $Y(\cdot, \omega)$ is oscillatory, so that the path $X(\cdot, \omega)$ is oscillatory, and hence, as Ω^* is an a.s. event, the solution of Equation (7) is a.s. oscillatory.

Observing that

$$\begin{aligned}
&\int_{t-\tau(t)}^{t} P(s)\, ds \\
&= \int_{t-\tau(t)}^{t} -b\ \exp(-(a - \frac{\sigma^2}{2})\tau(s))\ \exp(-\sigma(B(s) - B(s - \tau(s))))\, ds \\
&\geq -b\ \max(1, \exp(-(a - \frac{\sigma^2}{2})\overline{\tau})) \int_{t-\tau(t)}^{t} \exp(-\sigma(B(s) - B(s - \tau(s))))\, ds,
\end{aligned}$$

we see that, if

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} \exp(-\sigma(B(s) - B(s - \tau(s))))\, ds = \infty, \quad \text{a.s.}, \tag{18}$$

then Ω^* as prescribed above exists, and the theorem is proved. We state the validity of (18) in the following Lemma 1. The proof of this result is relegated to Appendix A. □

Figure 3 shows a solution and a sample trajectory of the solution of Equations (8) and (7) under the conditions of Proposition 5.

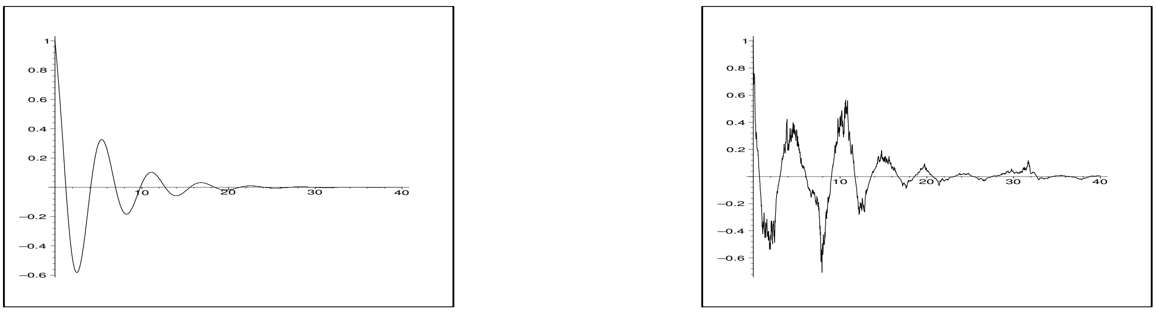

Figure 3: $dX(t) = \{0.35X(t) - X(t-1)\}dt + \sigma X(t)dW(t)$, $\psi \equiv 1$, $\sigma = 0$ (left) and $\sigma = 0.8$ (right).

The crucial condition in Proposition 5 which ensures oscillation is (18). We see that in the non-stochastic case (where $\sigma = 0$)

$$\int_{t-\tau(t)}^{t} \exp(-\sigma(B(s) - B(s - \tau(s))))\, ds = \tau(t),$$

and so (18) cannot hold, as τ is bounded. Moreover, if $\overline{\tau}$ is sufficiently small, the integral in (18) in the deterministic case may be sufficiently small so that the conditions in Proposition 2 hold. In this instance, the deterministic equation has a nonoscillatory solution: however, as Lemma 1 below reveals, (18) must always hold in the stochastic case where $\sigma \neq 0$, and so all solutions must be a.s. oscillatory.

Lemma 1. *Suppose that $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (11). If $\sigma \neq 0$ then*

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} \exp(-\sigma(B(s) - B(s - \tau(s))))\, ds = \infty, \quad a.s. \tag{19}$$

holds.

Remark 1. Consider the solution of (8), where τ satisfies (11) and (12). Then, letting $\phi(t) = e^{at}$, and $y(t) = x(t)/\phi(t)$ for $t \geq 0$, by analogy with (16), we have $y'(t) = -p(t)y(t - \tau(t))$ for $t > \overline{\tau}$, where $p(t) = -be^{-a\tau(s)}$. Thus, by Proposition 2 (with $g(t) = t - \tau(t)$) and $b < 0$, then (8) has a nonoscillatory solution for $a > 0$ when $-b\overline{\tau}e^{-a\underline{\tau}} < 1/e$, and for $a < 0$ when $-b\overline{\tau}e^{-a\overline{\tau}} < 1/e$. Hence Proposition 5 implies that the addition of a non-zero noise term to (8) to form the stochastic delay differential equation (7a) removes the possibility of a nonoscillatory solution. We provide an illustrative example in Figure 4.

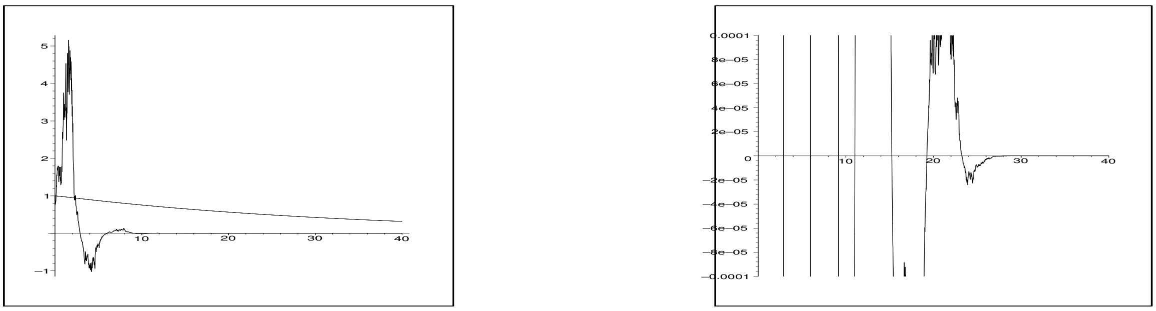

Figure 4: Left: $dX(t) = \{0.35X(t) - 1/e\, X(t-1)\}dt + \sigma X(t)dW(t)$, $\psi \equiv 1$, $\sigma = 0$ and $\sigma = 0.8$. Right: A zoom of the trajectory for $\sigma = 0.8$.

Remark 2. In the constant delay case ($\tau(t) = \tau$), we can see that not only do nonoscillatory solutions exist when $b < 0$, $-be^{-a\tau}e\tau < 1$, but that they are very prevalent. By Proposition 3 if the initial function ψ satisfies

$$\psi(0) + be^{-\lambda\tau} \int_{-\tau}^{0} e^{-\lambda s}\psi(s)\, ds \neq 0,$$

the solution of (8) with $\tau(t) = \tau$ and with initial function ψ is nonoscillatory. Thus, the set of initial functions in the deterministic case for which the solution is nonoscillatory is "large" for sufficiently small $b < 0$. In contrast, for the corresponding random system (7) with the same initial function ψ, the solution is a.s. oscillatory, by Proposition 5.

4. PROPERTIES OF THE ZERO SET

In this section, we consider the properties of the zero set of the process $(X(t))_{t\geq 0}$ given by (7). We give some motivating discussion, and then establish results on the topology of the zero set Z_X on almost all paths.

4.1. Discussion

The *zero set* of $(X(t))_{t\geq 0}$ given by (7) is defined by

$$Z_X = \{(t,\omega) \in \mathbb{R}^+ \times \Omega \ : \ X(t,\omega) = 0\}.$$

We concentrate on the zero set for fixed $\omega \in \Omega$, and show that this has the same topological structure for almost all $\omega \in \Omega$. To this end, we introduce

$$Z_X(\omega) = \{t \in \mathbb{R}^+ \ : \ X(t,\omega) = 0\}.$$

As almost all paths of X are non-differentiable almost everywhere, one might form the naive impression that the structure of Z_X would be very similar to that of the zero set of the standard Brownian motion $(B(t))_{t\geq 0}$ — for instance, in the terminology of this paper, B is an a.s. oscillatory process. Indeed, one might expect that the zeros of X would not be isolated, as those of Brownian motion are. The following result, found for example as Theorem 2.9.6 in Karatzas and Shreve, 1991, indicates some of the remarkable properties of the zero set of Brownian motion

$$Z_B = \{(t,\omega) \in \mathbb{R}^+ \times \Omega \ : \ B(t,\omega) = 0\}.$$

Proposition 6. *Define for fixed $\omega \in \Omega$ the zero set of $B(\cdot,\omega)$:*

$$Z_B(\omega) = \{0 \leq t < \infty \ : \ B(t,\omega) = 0\}.$$

Then for $\mathbb{P}$-a.e. $\omega \in \Omega$, the zero set $Z_B(\omega)$

(i) has Lebesgue measure zero,

(ii) is closed and unbounded,

(iii) has an accumulation point at $t = 0$*,*

(iv) has no isolated point in $(0, \infty)$*, and therefore*

(v) is dense in itself.

We think of Brownian motion as having typical sample path level sets, and in particular, sample path zero level set, of Itô processes. In the introduction, we observed that the nondelay version of (7) with multiplicative noise is strictly positive when it has a positive initial condition, so questions relating to its zero set do not arise. Instead, for instance, consider the version of (7) without delay and with additive noise given by

$$dX(t) = aX(t)\,dt + \sigma\,dB(t). \tag{20}$$

In the introduction, we noted that for $a < 0$ this process is a.s. oscillatory. When $X(0) = 0$, the stochastic differential equation has explicit solution

$$X(t) = e^{at} \int_0^t \sigma e^{-as}\,dB(s)$$

so the zero set of the path $X(\omega)$ coincides with the zero set of $M(\omega)$ where $M = \{M(t); \mathcal{F}_t^B; t \geq 0\}$ is the martingale given by

$$M(t) = \int_0^t e^{-as}\,dB(s).$$

This martingale has square variation

$$\langle M\rangle(t) = \int_0^t e^{-2as}\,ds = \frac{1}{-2a}(e^{-2at} - 1).$$

By the martingale time change theorem (see for example, Theorem 3.4.6 in Karatzas and Shreve, 1991, there exists a standard Brownian motion W such that $M(t) = W(\langle M\rangle(t))$. Therefore the zero set of M, and hence of X, the solution of (20), have the same properties as the zero set of a standard Brownian motion.

We also believe that the zero set of almost all sample paths of the additive noise version of (7) viz.,

$$dX(t) = (aX(t) + bX(t - \tau(t)))\,dt + \sigma\,dB(t) \tag{21}$$

has the same properties as the zero set of sample paths of Brownian motion, or the non-delay version of this equation, namely (20) above. Our belief is reinforced by Proposition 9 below, which shows that the zero set has an accumulation point at the time of the first zero.

In contrast to the complicated topology of Z_B, or the zero set of the solution of (20) or (21), consider the zero set of an arbitrary continuously differentiable and oscillatory function y defined on $\mathbb{R}^+$ which has zero set

$$Z_y = \{t \geq 0 \,:\, y(t) = 0\},$$

and suppose moreover that $y'(t) \neq 0$ for all $t \in Z_y$. Then, although properties (i), (ii) of Proposition 6 are satisfied for the zero set Z_y, property (iv) cannot hold, so all the elements of Z_y are isolated. This observation enables us to show that while the paths of X given by (7a) are nowhere differentiable, the zero set of X resembles that of y above, rather than that of B or the solutions of either (20) or (21). This holds, for example, if the initial function in (7b) is strictly positive.

4.2. Results on the zero set

We now present some results which give information on the structure of the zero set of X.

Proposition 7. *Let $X(\cdot,\omega)$ be a realisation of the process which is the solution of (7a) with initial function $\psi \in C([-\overline{\tau},0];\mathbb{R}^+)$, where the function τ satisfies (11). If, in addition, $b < 0$ and $\sigma \neq 0$, then for $\mathbb{P}$-a.e. $\omega \in \Omega$ the zero set $Z_X(\omega)$ has the following properties:*

(i) it has Lebesgue measure zero,

(ii) it is closed, unbounded and countable, and

(iii) every point of Z_X in $(0,\infty)$ is isolated.

Moreover, $Z_X(\omega) = \{t_n(\omega)\}_{n=1}^{\infty}$ where $\{t_n\}_{n\geq 1}$ is a nondecreasing sequence satisfying

$$t_1 > 0, \quad t_{n+1} > t_n + \underline{\tau} \quad a.s., \tag{22}$$

where $\underline{\tau} > 0$ is given by (11).

Proof. Note that the function Y which satisfies (17) obeys $Z_X(\omega) = Z_Y(\omega)$ for all $\omega \in \Omega$, so it suffices to study the zero set of Y. In the proof of Proposition 5, it was shown that Y is a.s. oscillatory, so the zero set Z_Y is unbounded for $\mathbb{P}$-a.e. $\omega \in \Omega$. Therefore we can define $t_1(\omega) = \inf\{t \geq 0 : Y(t,\omega) = 0\}$ for almost all $\omega \in \Omega$. By the continuity of Y and positivity of ψ, we have $t_1 > 0$. Furthermore, since $P(t) > 0$ for all $t \geq 0$, it follows from (17) that $Y'(t,\omega) < 0$ for all $t \in (0, t_1 + \underline{\tau})$. Since $Y(t_1) = 0$, we have $Y(t) < 0$ for all $t \in (t_1, t_1 + \tau)$, and indeed $Y(t_1 + \underline{\tau}) < 0$. By defining,

$$t_2(\omega) = \inf\{t \geq t_1 \,:\, Y(t_2,\omega) = 0\},$$

we see that $t_2(\omega) > t_1(\omega) + \underline{\tau}$. To prove relation (22), we proceed by an induction proof. Suppose we can define successively $t_n(\omega) = \inf\{t \geq t_{n-1} \ : \ Y(t,\omega) = 0\}$, for $n \geq 2$, and further suppose that $t_n > t_{n-1} + \underline{\tau}$. Without loss of generality, take $Y(t) > 0$ for $t \in (t_{n-1}, t_n)$. Then

$$Y(t) = -\int_{t_n}^{t} P(s)Y(s - \tau(s))\, ds < 0, \quad t \in (t_n, t_n + \underline{\tau}].$$

Therefore, as Y is continuous, we must have $t_{n+1}(\omega) > t_n(\omega) + \underline{\tau}$, which proves the induction hypothesis. Therefore $Z_Y = \{t_n > 0 \ : \ n \in \mathbb{N}\}$. The property of the sequence $(t_n)_{n\geq 1}$ establishes statements (i)-(iii) in the proposition for the zero set Z_Y, and therefore for the zero set Z_X. $\square$

Figures 5, 6 and 7 illustrate the behaviour concerning the discussion and propositions above of trajectories of the solution of (7a), (23) and (20), respectively. The same realisation of the Wiener process, i.e. the same set of increments, has been used for all three Figures. All pictures have been obtained with the Euler-Maruyama method and a step-size of 5/256.

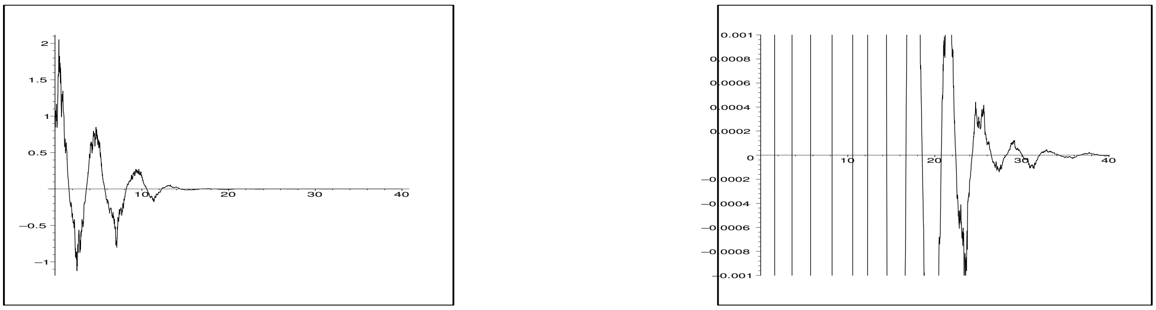

Figure 5: Left: $dX(t) = \{-0.35X(t) - X(t-1)\}dt + 0.8X(t)dW(t)$, $\psi \equiv 1$. Right: A zoom of the trajectory.

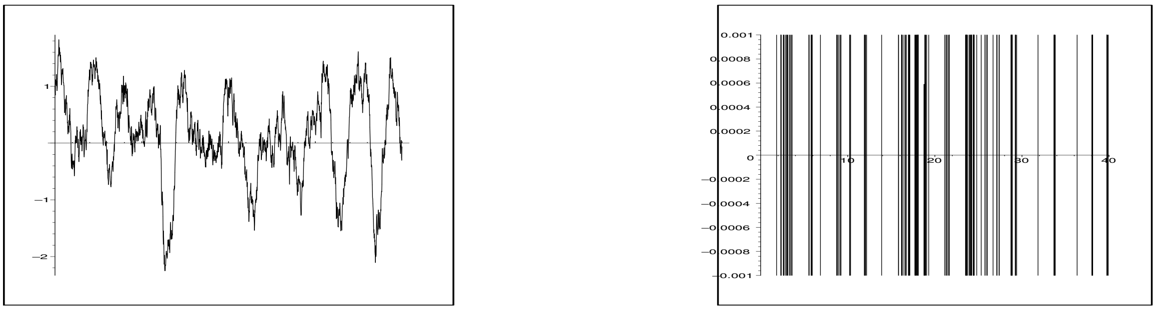

Figure 6: Left: $dX(t) = \{-0.35X(t) - X(t-1)\}dt + 0.8dW(t)$, $\psi \equiv 1$. Right: A zoom of the trajectory.

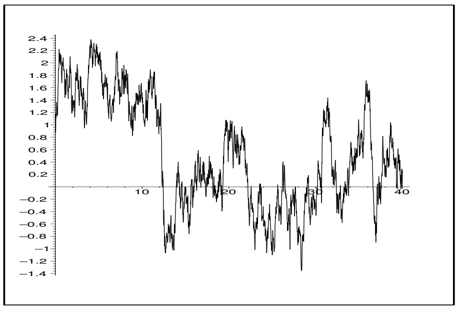

Figure 7: $dX(t) = -0.35X(t)dt + 0.8dW(t),\ X(0) = 1$.

There is one further point of note about the zero set Z_X which is interesting in the context of the non-differentiability of the path at almost all points: the process $(X(t))_{t\geq 0}$ is differentiable at $t = t^*$ if $t^* \in Z_X$. This largely explains the properties of the zero set of X.

Proposition 8. *Suppose the conditions of Proposition 7 are satisfied. Then there exists $\Omega^* \subset \Omega$ with $\mathbb{P}[\Omega^*] = 1$ such that for all $\omega \in \Omega^*$ and $t^* \in Z_X(\omega)$, we have that $t \mapsto X(t,\omega)$ is differentiable at $t = t^*$. Moreover, $X'(t^*) \neq 0$.*

Proof. Let $\Omega^* \subset \Omega$ be such that the solution of (7) is well–defined, and that the function $\Phi(\cdot,\omega)$ is continuous and strictly positive on $\mathbb{R}^+$, and the function $Y(\cdot,\omega)$ is continuously differentiable on $\mathbb{R}^+$, respectively. Suppose further that $X(\cdot,\omega)$ is oscillatory for $\omega \in \Omega^*$. Then Ω^* is an almost sure event.

Let $t^* \in Z_X(\omega)$ for $\omega \in \Omega^*$. Hereafter, we suppress ω-dependence. Using the fact that $X(t^*) = Y(t^*) = 0$, for $t \neq t^*$, we have

$$X(t) - X(t^*) = \Phi(t)Y(t) - \Phi(t^*)Y(t^*) = \Phi(t)Y(t) = \Phi(t)(Y(t) - Y(t^*)).$$

The continuity of $t \mapsto \Phi(t)$ and differentiability of $t \mapsto Y(t)$ on $\mathbb{R}^+$ yields

$$\lim_{t\to t^*} \frac{X(t) - X(t^*)}{t - t^*} = \lim_{t\to t^*} \Phi(t)\frac{Y(t) - Y(t^*)}{t - t^*} = \Phi(t^*)Y'(t^*).$$

Therefore $t \mapsto X(t,\omega)$ is differentiable at $t = t^*$. To show $X'(t^*) \neq 0$, suppose to the contrary $X'(t^*) = 0$: then $Y'(t^*) = 0$, as $\Phi(t) > 0$ for all $t \geq 0$. But this implies $\tilde{t} = t^* - \tau(t^*) \in Z_Y$, so $\tilde{t} \in Z_X$. Hence $\tilde{t} - t^* = \tau(t^*) < \underline{\tau}$. But this is in contradiction of Proposition 7, which states that the members of Z_X are separated by distances greater than $\underline{\tau}$.

$\square$

The differentiability of the path at a zero, and the fact that the zeros are isolated points are properties which are not shared by the additive noise delay-differential equation. We prove this in the following result.

Proposition 9. *Let τ be a positive and bounded function with $\tau(t) \leq \overline{\tau}$ and $\psi \in C([-\overline{\tau}, 0]; \mathbb{R})$ be a strictly positive function. Let $\sigma \neq 0$ and consider the equation*

$$dX(t) = (aX(t) + bX(t - \tau(t)))\,dt + \sigma\,dB(t) \tag{23a}$$

$$X(t) = \psi(t), \quad -\overline{\tau} \leq t \leq 0. \tag{23b}$$

Then the path $X(\cdot, \omega) : [0, \infty) \to \mathbb{R} : t \mapsto X(t, \omega)$ is nowhere differentiable for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Moreover, suppose $\theta = \inf\{t > 0 \,:\, X(t) = 0\}$ is an almost surely finite stopping time, and define the set

$$Z_X^\theta(\omega) = \{t \in [0, \infty) \,:\, X(\theta + t, \omega) = 0\}.$$

Then for $\mathbb{P}$-a.e. $\omega \in \Omega$.

(i) $Z_X^\theta(\omega)$ has an accumulation point at zero, so

(ii) $\theta(\omega) \in Z_X^\theta(\omega)$ is not an isolated member of $Z_X^\theta(\omega)$.

The proof of this result is sketched in Appendix A. The oscillation of solutions of such additive noise delay-differential equations is not the main focus of this paper: we present Proposition 9 in order to contrast the behaviour of the zero set of solutions of additive noise delay-differential equations with that of the solutions of multiplicative noise delay-differential equations.

5. CONCLUDING REMARKS

Certain modifications of these results are possible to include the effect of two or more constant delays, or non-autonomous equations, but no new ideas are involved. An interesting question which remains is whether the deterministic theory which exists to prove the oscillation of solutions of non-linear delay differential equations can be used in this framework to establish corresponding results for the non-linear stochastic system. A further open question is the extent to which the above linear theory admits a corresponding linearisation theory about the zero equilibrium.

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APPENDIX A

This Appendix contains a proof of the crucial technical Lemma 1, which was earlier deferred, and also the proof of Proposition 9, which we have relegated from the main text.

Proof of Lemma 1. If $\sigma < 0$, let $\tilde{\sigma} = -\sigma$. If $\sigma > 0$, note that $\tilde{B}(t) = -B(t)$ is also a standard Brownian motion, so it suffices to show, for $\tilde{\sigma} > 0$ that

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} \exp(\tilde{\sigma}(B(s) - B(s-\tau(s))))\, ds = \infty, \quad a.s.$$

Let $\tau_1 = \underline{\tau}/2$, so that $\tau(t) > \tau_1$, and define, for $t > \tau_1 + \overline{\tau}$ and $\alpha > 0$ the sets

$$C_{t,\alpha} = \{\omega \in \Omega \,:\, \min_{t-\tau_1 \le s \le t} B(s,\omega) - \max_{t-\tau_1-\overline{\tau} \le s \le t-\underline{\tau}} B(s,\omega) > \alpha\},$$

and

$$D_{t,\alpha} = \{\omega \in \Omega \,:\, B(s,\omega) - B(s-\tau(s),\omega) > \alpha \text{ for all } t-\tau_1 \le s \le t\}.$$

As τ satisfies (11), we have

$$\max_{t-\tau_1 \le t \le t} B(s-\tau(s),\omega) \le \max_{t-\tau_1-\overline{\tau} \le s \le t-\underline{\tau}} B(s,\omega).$$

Hence, for $\omega \in C_{t,\alpha}$ and $t-\tau_1 \le s \le t$, we have

$$B(s,\omega) - B(s-\tau(s),\omega) \ge \min_{t-\tau_1 \le s \le t} B(s,\omega) - \max_{t-\tau_1-\overline{\tau} \le s \le t-\underline{\tau}} B(s,\omega) > \alpha,$$

so that $\omega \in D_{t,\alpha}$. Thus $C_{t,\alpha} \subseteq D_{t,\alpha}$ for all $\alpha > 0$. Next define

$$U(t) = \int_{t-\tau(t)}^{t} \exp(\tilde{\sigma}(B(s) - B(s - \tau(s))))\, ds.$$

Then, for $\omega \in C_{t,\alpha}$, we have

$$U(t,\omega) \geq \int_{t-\tau_1}^{t} \exp(\tilde{\sigma}(B(s) - B(s-\tau(s))))\, ds \geq \int_{t-\tau_1}^{t} e^{\tilde{\sigma}\alpha}\, ds = \tau_1 e^{\tilde{\sigma}\alpha}.$$

Thus

$$\mathbb{P}[U(t) > \tau_1 e^{\tilde{\sigma}\alpha}] \geq \mathbb{P}[C_{t,\alpha}]. \tag{24}$$

Next, let a_n be an increasing sequence of positive numbers satisfying $a_{n+1} - a_n \geq \overline{\tau}$, and define $V_n = U(a_{2n})$. Since for $m \neq n$, V_n and V_m are functionals of increments of Brownian motion which are non-overlapping, it follows that V_n, $n = 1, 2, \dots$ is a sequence of independent, non-negative random variables. Thus, by the second Borel-Cantelli Lemma

$$\limsup_{n\to\infty} V_n = \infty, \quad a.s., \tag{25}$$

if and only if, for every $\beta > 0$

$$\sum_{n=1}^{\infty} \mathbb{P}[V_n > \beta] = \infty. \tag{26}$$

Therefore the assertion is proved if (26) (and in turn (25)) is proved, as

$$\limsup_{t\to\infty} U(t) \geq \limsup_{n\to\infty} U(a_{2n}) = \limsup_{n\to\infty} V_n = \infty, \quad a.s.$$

Moreover, with $\beta = \tau_1 e^{\tilde{\sigma}\alpha}$ by (24)

$$\mathbb{P}[V_n > \beta] \geq \mathbb{P}[C_{a_{2n},\alpha}],$$

so proving

$$\sum_{n=1}^{\infty} \mathbb{P}[C_{a_{2n},\alpha}] = \infty \tag{27}$$

establishes (26).

To show this, note now that we can write

$$\begin{aligned}
&\min_{a_{2n}-\tau_1 \leq s \leq a_{2n}} B(s) - \max_{a_{2n}-\tau_1-\overline{\tau} \leq s \leq a_{2n}-\underline{\tau}} B(s) \\
&\quad = \min_{a_{2n}-\tau_1 \leq s \leq a_{2n}} (B(s) - B(a_{2n} - \tau_1)) + (B(a_{2n} - \tau_1) - B(a_{2n} - \underline{\tau})) \\
&\qquad\qquad + \min_{a_{2n}-\tau_1-\overline{\tau} \leq a_{2n}-\underline{\tau}} (B(a_{2n} - \underline{\tau}) - B(s)),
\end{aligned}$$

and that the right hand side has the same distribution as the random variable

$$W = \min_{0\le s\le \tau_1} W^{(1)}(s) + W^{(2)}(\tau_1) + \min_{0\le s\le \overline{\tau}-\tau_1} W^{(3)}(s)$$

where $W^{(1)}$, $W^{(2)}$, $W^{(3)}$ are independent standard Brownian motions. Thus

$$\mathbb{P}[C_{a_{2n},\alpha}] = \mathbb{P}[W > \alpha],$$

so establishing $\mathbb{P}[W > \alpha] > 0$ for all $\alpha > 0$ is sufficient to prove (27), and hence the theorem. However, using the independence of the Brownian motions $W^{(1)}$, $W^{(2)}$, $W^{(3)}$, for any $\alpha > 0$, we have

$$\begin{aligned}
&\mathbb{P}[W > \alpha] \\
&\ge \mathbb{P}[\min_{0\le s\le\tau_1} W^{(1)}(s) > -\alpha/4, W^{(2)}(\tau_1) > 3\alpha/2, \min_{0\le s\le\overline{\tau}-\tau_1} W^{(3)}(s) > -\alpha/4] \\
&\ge \mathbb{P}[\min_{0\le s\le\tau_1} W^{(1)}(s) > -\alpha/4] \times \mathbb{P}[W^{(2)}(\tau_1) > 3\alpha/2] \\
&\qquad \times \mathbb{P}[\min_{0\le s\le\overline{\tau}-\tau_1} W^{(3)}(s) > -\alpha/4] \\
&> 0,
\end{aligned}$$

so we are done. □

We now turn to the proof of Proposition 9.

Proof of Proposition 9. Without loss, suppose $\sigma > 0$, and consider the continuous solution of (23), which is finite on $[0,\infty)$. To prove the a.s. nowhere differentiability of the paths of X, we appeal to a result of Dvoretzky (see for example, Karatzas and Shreve, 1991, Chapter 2.11, p.123) which says that there is a universal constant $c > 0$ such that

$$\limsup_{h\to 0^+} \frac{|B(t+h) - B(t)|}{\sqrt{h}} \ge c \quad \text{for all } t \ge 0, \text{ a.s.}$$

For every $t \ge 0$, $h > 0$, we have

$$\frac{X(t+h) - X(t)}{h} = \frac{1}{h}\int_t^{t+h} aX(s) + bX(s-\tau(s))\,ds + \sigma\frac{B(t+h) - B(t)}{h}.$$

The continuity of $t \mapsto X(t)$ along with Dvoretzky's result therefore yields

$$\limsup_{h\to 0^+} \frac{|X(t+h) - X(t)|}{h} = \infty \quad \text{for all } t \ge 0, \text{ a.s.}$$

so almost all paths are nowhere differentiable.

To prove the second part of the result, suppose $t > 0$. Since $X(\theta) = 0$

$$X(\theta + t) = \int_\theta^{\theta+t} aX(s) + bX(s-\tau(s))\,ds + \sigma(B(\theta+t) - B(\theta)).$$

Since ψ is deterministic, the natural filtration for B is also that for X. Therefore, θ is an almost surely finite stopping time for the natural filtration of B, where $B = \{B(t); \mathcal{F}_t; t \geq 0\}$. Then, by Theorem 2.6.16 in Karatzas and Shreve, 1991, the process $W = \{W(t); \mathcal{F}_t^W; t \geq 0\}$ given by $W(t) = B(\theta + t) - B(t)$ is a standard Brownian motion independent of $\mathcal{F}_{\theta+}$. Hence

$$\frac{X(\theta + t)}{t} = \frac{1}{t} \int_\theta^{\theta+t} aX(s) + bX(s - \tau(s))\, ds + \sigma \frac{W(t)}{t}.$$

where W is a standard Brownian motion independent of $\mathcal{F}_{\theta+}$. As $t \mapsto X(t)$ is continuous on $[-\overline{\tau}, \infty)$ almost surely, the Law of the Iterated Logarithm (see for example, Theorem 2.9.23 in Karatzas and Shreve, 1991) gives

$$\liminf_{t \to 0^+} \frac{X(t+\theta)}{t} = -\infty, \quad \limsup_{t \to 0^+} \frac{X(t+\theta)}{t} = \infty, \quad \text{a.s.} \tag{28}$$

Hereinafter, we restrict attention to the almost sure subset of Ω on which (28) holds and for which $\theta \in (0, \infty)$; we call this set $\tilde{\Omega}$. On $\tilde{\Omega}$ we may define $Y : [0, \infty) \to \mathbb{R} : t \mapsto Y(t) := X(t + \theta)$. Then if one can show that the event

$$\Omega^* = \{\omega \in \tilde{\Omega} \,:\, t \mapsto Y(t, \omega) \text{ has infinitely many zeros} \\ \text{in any time-interval } [0, \varepsilon], \text{ for any } \varepsilon > 0\}.$$

is almost sure, we have (i), and as a direct consequence, (ii). To prove that Ω^* is almost sure, it is enough to show that the event

$$\Omega_n = \{\omega \in \tilde{\Omega} \,:\, t \mapsto Y(t, \omega) \text{ has infinitely many zeros in } [0, 1/n]\}$$

is almost sure for all $n \in \mathbb{N}$, as $\Omega^* = \cap_{n=1}^\infty \Omega_n$. To do this, suppose there is an $n \in \mathbb{N}$ such that

$$\overline{\Omega}_n = \{\omega \,:\, t \mapsto Y(t, \omega) \text{ has finitely many zeros in } [0, 1/n]\}$$

is a set of positive probability. A contradiction to the fact that $\mathbb{P}[\overline{\Omega}_n] > 0$ yields that Ω_n is almost sure, and therefore proves the result. Now, let $\omega \in \overline{\Omega}_n$. The hypothesis implies that for each $\omega \in \overline{\Omega}_n$ there is a $T_n(\omega)$ with $0 < T_n(\omega) < 1/n$ such that $T_n(\omega) = \inf\{0 < t < 1/n \,:\, Y(t, \omega) = 0\}$. Hence either $Y(t, \omega) > 0$ for all $t \in (0, T_n(\omega))$ or $Y(t, \omega) < 0$ for all $t \in (0, T_n(\omega))$. Therefore, for all $t \in (0, T_n(\omega))$

$$\inf_{0<s<t} \frac{Y(s, \omega)}{s} > 0 \quad \text{or} \quad \sup_{0<s<t} \frac{Y(s, \omega)}{s} < 0,$$

so

$$\liminf_{t \to 0^+} \frac{Y(t, \omega)}{t} \geq 0 \quad \text{or} \quad \limsup_{t \to 0^+} \frac{Y(t, \omega)}{t} \leq 0$$

for all $\omega \in \Omega_n$. But as $\overline{\Omega}_n$ is assumed to be a set of positive probability, this is inconsistent with (28), and we have the desired contradiction. $\square$